TAFTCOLLEGE

Math 2140 Ordinary Differential Equations

Catalog Description:

The course is an introduction to ordinary differential equations including both quantitative and qualitative methods as well as applications from a variety of disciplines. It introduces the theoretical aspects of differential equations, including establishing when solution(s) exist, and techniques for obtaining solutions, including, series solutions, and singular points, Laplace transforms and linear systems. C-ID: MATH 240. Transfer Credit: CSU; UC.

SLO:

Course #1 - Model real situations using Ordinary Differential Equations.
Course #2 - Use appropriate methods to solve Ordinary Differential Equations such as LaPlace Transforms and Series Solutions.

Sample Problems:

Review for Differential Equations

Overview of Notation

- **Definitions:**
  - Summation:

$$\sum_{i=7}^{n} i = 7 + 8 + 9 + 10 + \cdots + (n-1) + n$$

  - Product:

$$\prod_{k=1}^{n} k = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot (n) \quad \text{(Note: In this case we get } n!\text{)}$$

  - $\mathbb{R}$ - Set of Real numbers
  - $\mathbb{Z}$ - set of integers $\{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$
  - $\mathbb{N}$ - set of natural numbers $\{0,1,2,3, \ldots\}$
  - $\mathbb{Q}$ - set of rational numbers $\left\{\frac{p}{q} \middle| \, p \text{ and } q \text{ are integers}\right\}$

- $\forall$ - "for all"
- $\exists$ - "There exists"
- $\Rightarrow$ - implies
- $\Leftrightarrow$ - if and only if: $A \Leftrightarrow B.\ (If\ A\ then\ B, and\ if\ B\ then\ A)$
- ϵ – "is an element of"
- $\therefore$ - Therefore
- $\cup$ - Union
- $\cap$ - Intersection
- $\emptyset$ - Null Set, Empty Set

1. **Solve the following equations for y.**
   a. $2xy + x^2 + sin(x) = 3y - 4xy + 2$

   b. $sin(y) + 5 = x^2 + 3x + 2$

   c. $\log_{17} y = x + 2$

   d. $\frac{1}{y+2} + x^2 = cos(x)$

   e. $(x + 2y)^2 - 3x + sin(0) = 4y^2 + 2xy$

2. **Evaluate the following limits**

a. $\lim_{t \to \infty}(5)$

b. $\lim_{t \to \infty}(5e^{-2t})$

c. $\lim_{t \to \infty}(5\cos(t)\, e^{-2t})$

d. Assume $a, k, \&\ Q_0$ are arbitrary constants. Find $\lim_{t \to \infty}\left(\frac{a}{k} + \left(Q_0 - \frac{a}{k}\right)e^{-kt}\right)$

e. Assume k, T_0 & T_m are arbitrary constants. Find $\lim_{t\to\infty}\left(T_m + (T_0 - T_m)e^{-kt}\right)$

3. Evaluate the following integrals

a. $\int \sec^2(x) + 2 + \sin(4) + x^3 dx$

b. $\int 4x^2 + 5x^{-3} + \sin\left(\frac{x}{2}\right) dx$

c. $\int \sqrt{s} + \sqrt[3]{s} + \sqrt[4]{s} ds$

d. $\int \sin^4(t)\cos^3(\text{t})\,\text{dt}$

e. $\int_{-1}^{2} 6t(t^2-1)^2 dt$

f. $\int x^2 e^x dx$

g. $\int \tan(t) + \ln(t)\, dt$

h. $\int (x^2y^3 + \sin(xy))dy$

i. $\int \frac{3x+11}{x^2-x-6}dx$

4. Let $f(x, y, z) = x^2y^3 + \cos(xz) + e^{xyz}$. Find the following

a. $f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y^3 + \cos(xz) + e^{xyz})$

b. $f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y^3 + \cos(xz) + e^{xyz})$

c. $f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$

d. $f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$

5. Find the following derivatives

a. $\frac{d}{dx}\left(e^{x^2} + \ln(x) + \sin(x)\right)$

b. $\frac{\partial}{\partial t}\left(8st^2 + \cos\left(\frac{t}{7}\right) - \tan(st)\right)$

c. $\frac{d}{d\theta}(\sin(cos(2\theta)))$

Solutions:

1. Solve the following equations for y.

a. $2xy + x^2 + sin(x) = 3y - 4xy + 2$

$$\leftrightarrow 2xy - 3y + 4xy = 2 - x^2 + \sin(x)$$

$$\leftrightarrow y(2x - 3 + 4x) = 2 - x^2 + \sin(x)$$

$$\rightarrow y = \frac{2 - x^2 + \sin(x)}{6x - 3}$$

b. $sin(y) + 5 = x^2 + 3x + 2$

$$\leftrightarrow \sin(y) = x^2 + 3x - 3$$

$$\rightarrow y = \sin^{-1}(x^2 + 3x - 3)$$

c. $\log_{17} y = x + 2$

$$\rightarrow y = 17^{x+2}$$

d. $\frac{1}{y+2} + x^2 = cos(x)$

$$\leftrightarrow 1 + x^2(y + 2) = \cos(x)\,(y + 2)$$

$$\leftrightarrow 1 + x^2y + 2x^2 = ycos(x) + 2\cos(x)$$

$$\leftrightarrow x^2y - ycos(x) = 2\cos(x) - 1 - 2x^2$$

$$\rightarrow y = \frac{2\cos(x) - 1 - 2x^2}{x^2 - \cos(x)}$$

e. $(x+2y)^2 - 3x + sin(0) = 4y^2 + 2xy$

$$\leftrightarrow x^2 + 4xy + 4y^2 - 3x + 0 = 4y^2 + 2xy$$

$$\leftrightarrow x^2 + 2xy - 3x = 0$$

$$\rightarrow y = \frac{3x - x^2}{2x} = \frac{3-x}{2}$$

2. Evaluate the following limits

a. $\lim_{t\to\infty}(5) = 5$

b. $\lim_{t\to\infty}(5e^{-2t}) = \lim_{t\to\infty}\left(\frac{5}{e^{2t}}\right) = 0$

c. $\lim_{t\to\infty}(5\cos(t)\, e^{-2t}) = 5 \cdot \lim_{t\to\infty}\left(\frac{\cos(t)}{e^{2t}}\right) \rightarrow$ *Apply the Squeeze Theorem*

$$Since\ -1 \le \cos(t) \le 1 \rightarrow \frac{-1}{e^{2t}} \le \frac{\cos(t)}{e^{2t}} \le \frac{1}{e^{2t}}$$

$$\rightarrow \lim_{t\to\infty}\frac{-1}{e^{2t}} \le \lim_{t\to\infty}\frac{\cos(t)}{e^{2t}} \le \lim_{t\to\infty}\frac{1}{e^{2t}}$$

$$\rightarrow 0 \le \lim_{t\to\infty}\frac{\cos(t)}{e^{2t}} \le 0$$

$$\therefore \lim_{t\to\infty}(5\cos(t)\, e^{-2t}) = 0$$

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d. Assume $a, k, \& \ Q_0$ are arbitrary constants where $k \neq 0$. Find $\lim_{t\to\infty}\left(\frac{a}{k} + \left(Q_0 - \frac{a}{k}\right)e^{-kt}\right)$

$$\lim_{t\to\infty}\left(\frac{a}{k} + \left(Q_0 - \frac{a}{k}\right)e^{-kt}\right) = \lim_{t\to\infty}\left(\frac{a}{k}\right) + \left(Q_0 - \frac{a}{k}\right) \cdot \lim_{t\to\infty}\left(e^{-kt}\right)$$

Note: We can see that the limit above depends on the value of k. So we split it up into 2 cases: $k < 0 \ \& \ k > 0$.

$$\text{Case 1:} \ If\ k < 0 \rightarrow \lim_{t\to\infty}\left(\frac{a}{k}\right) + \left(Q_0 - \frac{a}{k}\right) \cdot \lim_{t\to\infty}\left(e^{-kt}\right) = \pm\infty$$

$$\text{Case 2:} \ If\ k > 0 \rightarrow \lim_{t\to\infty}\left(\frac{a}{k}\right) + \left(Q_0 - \frac{a}{k}\right) \cdot \lim_{t\to\infty}\left(e^{-kt}\right) = \frac{a}{k}$$

e. Assume $k, T_0 \ \& \ T_m$ are nonzero arbitrary constants where $k \neq 0$. Find $\lim_{t\to\infty}\left(T_m + (T_0 - T_m)e^{-kt}\right)$

$$\lim_{t\to\infty}\left(T_m + (T_0 - T_m)e^{-kt}\right) = \lim_{t\to\infty}(T_m) + (T_0 - T_m) \cdot \lim_{t\to\infty}\left(e^{-kt}\right)$$

Note: We can see that the limit above depends on the value of k. So we split it up into 2 cases: $k < 0 \ \& \ k > 0$.

$$\text{Case 1:} \ If\ k < 0 \rightarrow \lim_{t\to\infty}(T_m) + (T_0 - T_m) \cdot \lim_{t\to\infty}\left(e^{-kt}\right) = \pm\infty$$

$$\text{Case 2:} \ If\ k > 0 \rightarrow \lim_{t\to\infty}(T_m) + (T_0 - T_m) \cdot \lim_{t\to\infty}\left(e^{-kt}\right) = T_m$$

3. Evaluate the following integrals

a. $\int \sec^2(x) + 2 + \sin(4) + x^3 dx = \tan(x) + 2x + \sin(4)\,x + \frac{1}{4}x^4 + C$

$$Verify: \frac{d}{dx}\left(\tan(x) + 2x + \sin(4)\,x + \frac{1}{4}x^4 + C\right) = sec^2(x) + 2 + \sin(4) + x^3$$

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b. $\int 4x^2 + 5x^{-3} + \sin\left(\frac{x}{2}\right) dx = \frac{4}{3}x^3 - \frac{5}{2}x^{-2} - 2\cos\left(\frac{x}{2}\right) + C$

$$Verify: \frac{d}{dx}\left(\frac{4}{3}x^3 - \frac{5}{2}x^{-2} - 2\cos\left(\frac{x}{2}\right) + C\right) = 4x^2 + 5x^{-3} + \sin\left(\frac{x}{2}\right)$$

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c. $\int \sqrt{s} + \sqrt[3]{s} + \sqrt[4]{s} ds = \frac{2}{3}s^{3/2} + \frac{3}{4}s^{4/3} + \frac{4}{5}s^{5/4} + C$

$$Verify: \frac{d}{ds}\left(\frac{2}{3}s^{3/2} + \frac{3}{4}s^{4/3} + \frac{4}{5}s^{5/4} + C\right) = \sqrt{s} + \sqrt[3]{s} + \sqrt[4]{s}$$

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d. $\int \sin^4(t)\cos^3(t)\,dt = \int sin^4(t)cos(t)cos^2(t)dt = \int sin^4(t)cos(t)(1 - sin^2(t))dt$

$$\leftrightarrow \int sin^4(t)cos(t)dt - \int sin^6(t)cos(t)dt \rightarrow Let\ u = sin(t)\ then\ du = cos(t)dt$$

$$\leftrightarrow \int u^4du - \int u^6du = \frac{1}{5}u^5 - \frac{1}{7}u^7 + C$$

$$(Substitute\ back) \leftrightarrow \frac{1}{5}sin^5(t) - \frac{1}{7}sin^7(t) + C$$

$$Verify: \frac{d}{dt}\left(\frac{1}{5}sin^5(t) - \frac{1}{7}sin^7(t) + C\right) = \sin^4(t)\cos^3(t)$$

e. $\int_{-1}^{2} 6t(t^2 - 1)^2 dt \rightarrow Let\ u = (t^2 - 1)\ then\ du = 2tdt \rightarrow 3du = 6tdt$

$$\leftrightarrow \int_{LB}^{UB} 3u^2du = u^3|_{u=LB}^{u=UB}$$

$$(Substitute\ back) \leftrightarrow (t^2 - 1)^3|_{t=-1}^{t=2} \rightarrow (3)^3 - (0)^3 = 27$$

Note: Instead of having to substitute back before evaluating our definite integral at the lower and upper bounds, we could solve for our bounds in terms of u using the substitution that we have already established.

$$Since\ u = t^2 - 1 \rightarrow UB = (2)^2 - 1 = 3 \quad \& \quad LB = (-1)^2 - 1 = 0$$

Then we end up with the following definite integral in terms of u:

$$\int_0^3 3u^2du = u^3|_{u=0}^{u=3} \rightarrow (3)^3 - (0)^3 = 27$$

Whether we substitute the bounds or substitute our answer back before plugging in the bounds, we will always end up with the exact same solution.

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f. $\int xe^x dx$

$$\text{Use Integration by Parts} \rightarrow \int u'v = uv - \int uv'$$

$$\rightarrow u = e^x \;\&\; u' = du = e^x \quad \text{and} \quad v = x \;\&\; v' = dv = 1$$

$$\rightarrow e^x \cdot x - \int e^x \cdot 1dx = xe^x - e^x + C$$

Note: You can also use Tabular Method to solve this:

Derivative	Integral
x	e^x
1	e^x
0	e^x

$$\rightarrow +(x \cdot e^x) - (1 \cdot e^x) + C$$

$$\rightarrow xe^x - e^x + C$$

g. $\int \tan(t) + \ln(t)\, dt = \int \frac{\sin(t)}{\cos(t)} dt + \int 1 \cdot \ln(t)\, dt$

$$\text{Use U} - \text{Substitution \& Integration by Parts}$$

$$\int \frac{\sin(t)}{\cos(t)} dt \rightarrow Let\ u = cos(t)\ then\ du = -sin(t)dt$$

$$\leftrightarrow \int -\frac{1}{u} du = -ln|u| + C_1 = -ln|cos(t)| + C_1 = \ln|sec(t)| + C_1$$

$$\int 1 \cdot \ln(t)\, dt \rightarrow u = t \;\&\; u' = du = 1 \quad \text{and} \quad v = \ln(t) \;\&\; v' = dv = \frac{1}{t}$$

$$\rightarrow t \cdot \ln(t) - \int t \cdot \frac{1}{t} dt = t \cdot \ln(t) - t + C_2$$

$$\therefore \int \frac{\sin(t)}{\cos(t)} dt + \int 1 \cdot \ln(t)\, dt = ln|sec(t)| + t \cdot \ln(t) - t + C$$

h. $\int (x^2y^3 + \sin(xy))dy$

Note: We are treating x as a constant in this example

$$\rightarrow x^2 \int y^3 dy + \int \sin(xy)\, dy = x^2\left(\frac{1}{4}y^4\right) - \frac{1}{x}\cos(xy) + C$$

$i. \int \frac{3x+11}{x^2-x-6}dx$

Use Partial Fraction Decomposition

$$\frac{3x+11}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2} \rightarrow \text{Multiply both sides of equation by LCD: } (x-3)(x+2)$$

$$\rightarrow 3x + 11 = A(x+2) + (Bx-3)$$

Note: Now we need to set our coefficients equal and solve for A & B.

$$\rightarrow A + B = 3 \;\;\&\;\; 2A - 3B = 11$$

$$\rightarrow A = 4 \;\&\; B = -1$$

$$\int \frac{3x+11}{x^2-x-6}dx = \int \frac{4}{x-3}dx - \int \frac{1}{x+2}dx = 4\ln|x-3| - \ln|x+2| + C$$

4. Let $f(x,y,z) = x^2y^3 + \cos(xz) + e^{xyz}$. Find the following

a. $f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y^3 + \cos(xz) + e^{xyz})$

$$f_x = 2xy^3 - z \cdot sin(xz) + yz \cdot e^{xyz}$$

b. $f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y^3 + \cos(xz) + e^{xyz})$

$$f_y = 3x^2y^2 + xze^{xyz}$$

c. $f_{yx} = \left(f_y\right)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$

From part (b) we know that $f_y = 3x^2y^2 + xze^{xyz}$, so now we need: $\frac{\partial}{\partial x}(3x^2y^2 + xze^{xyz})$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 + xze^{xyz}) = 6xy^2 + xyz^2e^{xyz} + ze^{xyz}$$

d. $f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$

From part (a) we know that $f_x = 2xy^3 - z \cdot sin(xz) + yz \cdot e^{xyz}$, so now we need: $\frac{\partial}{\partial x}(2xy^3 - z \cdot sin(xz) + yz \cdot e^{xyz})$

$$f_{xx} = \frac{\partial}{\partial x}(2xy^3 - z \cdot sin(xz) + yz \cdot e^{xyz}) = 2y^x - z^2\cos(xz) + y^2z^2e^{\boldsymbol{xyz}}$$

5. Find the following derivatives

a. $\frac{d}{dx}\left(e^{x^2} + \ln(x) + \sin(x)\right)$

$$\rightarrow \frac{d}{dx}\left(e^{x^2}\right) + \frac{d}{dx}(\ln(x)) + \frac{d}{dx}(\sin(x)) = 2xe^{x^2} + \frac{1}{x} + \cos(x)$$

b. $\frac{\partial}{\partial t}\left(8st^2 + \cos\left(\frac{t}{7}\right) - \tan(st)\right)$

$$\rightarrow \frac{\partial}{\partial t}(8st^2) + \frac{\partial}{\partial t}\left(\cos\left(\frac{t}{7}\right)\right) - \frac{\partial}{\partial t}(\tan(st)) = 16st - \frac{1}{7}\sin\left(\frac{t}{7}\right) - s \cdot \sec^2(st)$$

c. $\frac{d}{d\theta}(\sin(cos(2\theta)))$

Use Chain Rule $\rightarrow \cos(cos(2\theta)) \cdot (-\sin(2\theta)) \cdot 2 = -2 \cdot \sin(2\theta) \cdot \cos(cos(2\theta))$